WESS-ZUMINO-WITTEN MODEL ON ELLIPTIC CURVES AT THE CRITICAL LEVEL

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Abstract. We construct a Gaudin type lattice model as the Wess-Zumino-Witten model on elliptic curves at the critical level. Bethe eigenvectors are obtained by the bosonisation technique.

0. Introduction

The goal of this article is to construct a lattice model which is a variant of the Gaudin model with the help of the Wess-Zumino-Witten (WZW) model on elliptic curves at the critical level and to find its eigenvectors by means of the bosonisation of the WZW model.

As is well known, correlation functions of the WZW model on the Riemann sphere satisfy the Knizhnik-Zamolodchikov equations when the level k of the model is not critical, i.e., $k \neq -h^\vee$, where $h^\vee$ is the dual Coxeter number of the simple Lie algebra $\mathfrak{g}$ which describes symmetry of the model. In constrast to this case, when the level is critical, $k = -h^\vee$, there can be no longer such an equation since the Sugawara construction of the energy-momentum tensor breaks down.

In [FFR] Feigin, Frenkel and Reshetikhin found an interpretation of such case as a lattice model called the (rational) Gaudin model [G1], [G2], [G3], a quasi-classical limit of the totally isotropic spin chain model (the XXX model). (See [S].) It is also shown in [FFR] that the free field realisation of the WZW model provides a new diagonalisation method which recovers the results of the Bethe Ansatz method.

In this paper we apply this story to the WZW model on elliptic curves. The state space of the lattice model thus obtained is a space of functions over the Cartan subalgebra. The transfer matrix (the generating function of the Hamiltonians) is a quasi-classical limit of the IRF type lattice model. Therefore we name the model a "face type Gaudin model". The bosonisation technique is also applied to find a Bethe Ansatz type eigenvectors.

Felder and Varchenko [FV1], [FV2] studied this system and its Bethe Ansatz, which arose from the stationary phase method of the integral representation of solutions of the KZB equations. Their results are used in the analysis of a spin chain model with elliptic exchanges in [I]. The same kind of system (the Gaudin-Calogero model) has been also studied by Enriquez, Feigin and Rubtsov [ER], [EFR] who started from the quantisation of the Hitchin system on elliptic curves.

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This paper is organised as follows. In Section 1 we state our main results. Section 2 explains how to derive our transfer matrix from the WZW model on elliptic curves at the critical level. In order to construct the Bethe eigenvectors we make use of the Wakimoto modules with the critical level, which we recall in Section 3. The last section, Section 4, shows that the free field theory gives the eigenvector of the transfer matrix in the form of the Bethe vector.

Details shall be published in a forthcoming paper. Mathematically delicate conditions like finiteness of modules are not specified unless they are essential.

1. Main results

First we fix notations. Throughout this paper τ is a complex number with positive imaginary part. The elliptic curve with modulus τ is denoted by $X = X_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of rank l, $\mathfrak{h}$ be its Cartan subalgebra and

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha \tag{1.1}$$

be the root space decomposition, where Δ is the set of roots. We use the Cartan-Killing form normalised as follows:

$$(A \,|\, B) := \frac{1}{2h^\vee} \operatorname{Tr}_{\mathfrak{g}}(\operatorname{ad} A \operatorname{ad} B) \in \mathbb{C} \text{ for } A, B \in \mathfrak{g}, \tag{1.2}$$

We identify $\mathfrak{h}$ and its dual space $\mathfrak{h}^*$ through this inner product. We fix the simple roots $\{\alpha_1, \dots, \alpha_l\}$, Chevalley generators $\{H_i, E_i, F_i\}_{i=1,\dots,l}$ and a basis e_α of $\mathfrak{g}_\alpha$, such that $e_{\alpha_i} = E_i$ for $i = 1, \dots, l$ and $(e_\alpha | e_{-\alpha'}) = \delta_{\alpha,\alpha'}$. The set of positive and negative roots are denoted by $\Delta_+ = \{\beta_1, \dots, \beta_s\}$ and Δ_- respectively. We fix an orthonormal basis $\{h_r\}_{r=1,\dots,l}$ of $\mathfrak{h}$ and the coordinate system of $\mathfrak{h}$, $(\xi_1, \dots, \xi_l) \in \mathbb{C}^l$, associated to it.

Let V_i $(i = 1, \dots, N)$ be finite dimensional irreducible representations of $\mathfrak{g}$ with the highest weight λ_i and $V(0)$ be the 0-weight space of $V = V_1 \otimes \cdots \otimes V_N$. We denote the action of $\mathfrak{g}$ on the i-th factor V_i by ρ_i as usual. The dual (right) action of $\mathfrak{g}$ on $\Phi \in V^* = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is denoted as

$$\rho_i^*(A)\Phi(v) := \Phi(\rho_i(A)v), \tag{1.3}$$

and the 0-weight space of V^* by $V^*(0)$.

We define a differential operator $\hat{\tau}(u)$ on $V^*(0)$-valued functions on

$$S = \{H \in \mathfrak{h} \mid \alpha(H) \notin \mathbb{Z} \text{ for all } \alpha \in \Delta\} \tag{1.4}$$

with fixed complex parameters z_i $(i = 1, \dots, N)$ and a spectral parameter u as follows:

$$\begin{aligned} \hat{\tau}(u) &:= \frac{1}{2} \sum_{r=1}^{l} \nabla^2_{\xi_r,u} + \\ &+ \frac{1}{2} \sum_{i,j=1}^{N} \sum_{\alpha\in\Delta} w_{\alpha(H)}(z_i - u)\, w_{-\alpha(H)}(z_j - u) \rho_j^*(e_{-\alpha}) \rho_i^*(e_\alpha), \end{aligned} \tag{1.5}$$

where

$$\nabla_{\xi_r,u} := \frac{\partial}{\partial \xi_r} - \sum_{i=1}^{N} \zeta_{11}(z_i - u)\rho_i^*(h_r). \tag{1.6}$$

The quasi-periodic functions $w_c(z)$ and $\zeta_{11}(z)$ are defined by (A.2).

Since this operator can be interpreted as a trace of square of the dynamical (or modified) classical r-matrix which is a classical limit of the IRF type lattice models (cf. [F1], [F2], [FS]), we call $\hat\tau(u)$ the *transfer matrix of the face type Gaudin model.*

Theorem 1.1. *The operator $\hat\tau(u)$ commutes with itself:*

$$[\hat\tau(u), \hat\tau(u')] = 0. \tag{1.7}$$

This can be checked by direct computation, but we shall show in Section 2 that it can be proved with the help of the WZW model at the critical level.

Taking Theorem 1.1 into account, we can pose a question of simultaneous diagonalisation of $\hat\tau(u)$. Our main result is the following Bethe Ansatz solution of this problem. Assume that V_i is the dual Verma module $M^*_{\lambda_i}$ of $\mathfrak{g}$ and that the sum of the weights λ_i belongs to the positive root lattice:

$$\sum_{i=1}^{N} \lambda_i = \sum_{j=1}^{M} \alpha_{i(j)}, \tag{1.8}$$

for a sequence $\{\alpha_{i(j)}\}_{j=1,\dots,M}$ of simple roots. The symbol $\jmath(v)$ for $v \in M^*_\lambda$ is the canonical paring of v with the highest weight vector of the Verma module M_λ.

Theorem 1.2. *If there are complex numbers t_j $(j = 1, \dots, M)$ satisfying the* Bethe Ansatz equation*,*

$$\sum_{i=1}^{N} (\alpha_{i(j)} \mid \lambda_i)\zeta_{11}(t_j - z_i) = \sum_{j' \neq j} (\alpha_{i(j)} \mid \alpha_{i(j')})\zeta_{11}(t_j - t_{j'}), \tag{1.9}$$

for any $j = 1, \dots, M$, then

$$\Psi(H; v) := \sum_{\{I_j\}} \prod_{a=1}^{N} \langle I_a; v_a; z_a, t_j(j \in I_a)\rangle \tag{1.10}$$

is an eigenvector of $\hat\tau(u)$. Here $\{I_j\}$ is a partition of the set $\{1, \dots, M\}$ into N sets, $I_1 \sqcup \cdots \sqcup I_N = \{1, \dots, M\}$, and the symbol $\langle I; v; t_j(j \in I)\rangle$ is defined as follows:

(1.11)
$$\begin{aligned}\langle I; v; z, t_j(j \in I)\rangle := \sum_{\sigma \in \mathfrak{S}} w_{\alpha_{i(\sigma(1))}}(t_{\sigma(1)} - t_{\sigma(2)})\, w_{\alpha_{i(\sigma(1))}+\alpha_{i(\sigma(2))}}(t_{\sigma(2)} - t_{\sigma(3)}) \times \cdots \\ \cdots \times w_{\alpha_{i(\sigma(1))}+\alpha_{i(\sigma(2))}+\cdots+\alpha_{i(\sigma(m))}}(t_{\sigma(m)} - z)\, \jmath(E_{i(\sigma(m))} \cdots E_{i(\sigma(1))}v)\end{aligned}$$

if $I = \{1, \dots, m\}$. (We write w_α instead of $w_{\alpha(H)}$ for short.) The eigenvalue of $\Psi(H; v)$ is

$$\tau_\Psi(u) := \frac{1}{2} \sum_{r=1}^{l} \boldsymbol{\zeta}(h_r; z, t; u)^2 + \frac{\partial}{\partial u} \boldsymbol{\zeta}(\rho; z, t; u). \tag{1.12}$$

Here $\boldsymbol{\zeta}$ is defined by

$$\boldsymbol{\zeta}(h;z,t;u) := \sum_{i=1}^{N} \lambda_i(h)\zeta_{11}(z_i - u) - \sum_{j=1}^{M} \alpha_{i(j)}(h)\zeta_{11}(t_j - u), \tag{1.13}$$

and ρ is the half sum of the positive roots of $\mathfrak{g}$.

We shall show how to prove this theorem by bosonisation in Section 4.

2. WZW model on elliptic curves at the critical level

The idea behind the definition (1.5) and Theorem 1.1 is that the linear functional $\Phi(H;v)$ and $(\hat{\tau}(u)\Phi)(H;v)$ are analogues of an N-point function of the WZW model and the correlation funcion of the energy-momentum tensor, respectively.

First we define the geometric data on which the WZW model lives. Let $\mathfrak{X} = S \times X_\tau$ be the trivial family of elliptic curves and $\pi_{\mathfrak{X}/S} : \mathfrak{X} \to S$ be the projection. The divisor of $\mathfrak{X}$ corresponding to the parameter z_i is denoted by P_i and their sum by D:

$$P_i := S \times z_i \bmod \mathbb{Z} + \mathbb{Z}\tau \subset \mathfrak{X}, \qquad D := P_1 + \cdots + P_N \tag{2.1}$$

The following general definitions of N-point functions is valid not only for the WZW model but also for the free field theories which we use later.

Assume that the following Lie algebraic data on $\mathfrak{X}$ are given:

- $\mathfrak{a}_{\mathfrak{X}}$: a Lie algebra bundle with a fibre isomorphic to a Lie algebra $\mathfrak{a}$.
- $\langle \cdot \mid \cdot \rangle : \mathfrak{a}_{\mathfrak{X}} \times_{\mathcal{O}_S} \mathfrak{a}_{\mathfrak{X}} \to \Omega^1_{\mathfrak{X}/S}$: an $\mathcal{O}_S$-bilinear $\Omega^1_{\mathfrak{X}/S}$-valued pairing which is a 2-cocycle up to exact forms:

$$\begin{gathered} \mathfrak{a}_{\mathfrak{X}} \times \mathfrak{a}_{\mathfrak{X}} \ni (A,B) \mapsto \langle A \mid B \rangle \in \Omega^1_{\mathfrak{X}/S}, \\ \langle A \mid B \rangle + \langle B \mid A \rangle \in d_{\mathfrak{X}/S}\mathcal{O}_{\mathfrak{X}}, \\ \langle [A,B] \mid C \rangle + \langle [B,C] \mid A \rangle + \langle [C,A] \mid B \rangle \in d_{\mathfrak{X}/S}\mathcal{O}_{\mathfrak{X}}, \end{gathered} \tag{2.2}$$

  for any $A, B, C \in \mathfrak{a}_{\mathfrak{X}}$. For example, if there is a connection ∇ on $\mathfrak{a}_{\mathfrak{X}}$ along the fibre compatible with the Lie algebra structure,

$$\begin{gathered} \nabla : \mathfrak{a}_{\mathfrak{X}} \to \mathfrak{a}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/S}, \\ \nabla[A,B] = [\nabla A, B] + [A, \nabla B] \text{ for } A, B \in \mathfrak{a}_{\mathfrak{X}}, \end{gathered} \tag{2.3}$$

  and an invariant $\mathcal{O}_{\mathfrak{X}}$-inner product $(\cdot \mid \cdot)$ of $\mathfrak{a}_{\mathfrak{X}}$ which satisfies

$$d_{\mathfrak{X}/S}(A|B) = (\nabla A \mid B) + (A \mid \nabla B) \in \Omega^1_{\mathfrak{X}/S} \quad \text{for } A, B \in \mathfrak{a}_{\mathfrak{X}}, \tag{2.4}$$

  then $\langle A \mid B \rangle := (\nabla A \mid B)$ has desired properties.

Example 2.1. In this section we use the following data. The Lie algebra bundle $\mathfrak{g}_{\mathfrak{X}}$ over $\mathfrak{X}$ is defined as the quotient of $S \times \mathbb{C} \times \mathfrak{g}$ by the $\mathbb{Z}^2$-action,

$$(m,n) \cdot (H;t;A) = (H; t + m\tau + n; e^{2\pi i m \operatorname{ad} H} A), \tag{2.5}$$

for $(m,n) \in \mathbb{Z}^2$ and $(H;t;A) \in S \times \mathbb{C} \times \mathfrak{g}$. The connection ∇ of (2.3) is the trivial differentiation $d/dt \otimes dt$ along the elliptic curve and the invariant bilinear form $(\cdot \mid \cdot)$ is defined by (1.2).

For $i = 1, \dots, N$, we put

$$\begin{aligned} \mathfrak{a}_S^{P_i} &:= (\pi_{\mathfrak{X}/S})_*(\mathfrak{a}_{\mathfrak{X}}(*P_i))^{\wedge}_{P_i} \cong (\mathfrak{a} \otimes \mathcal{O}_S)((x_i)), \\ \mathfrak{a}_{S,+}^{P_i} &:= (\pi_{\mathfrak{X}/S})_*(\mathfrak{a}_{\mathfrak{X}})^{\wedge}_{P_i} \cong (\mathfrak{a} \otimes \mathcal{O}_S)[[x_i]], \\ \mathfrak{a}_S^D &:= \bigoplus_{i=1}^N \mathfrak{a}_S^{P_i}, \qquad \hat{\mathfrak{a}}_S^D := \mathfrak{a}_S^D \oplus \mathcal{O}_S \hat{k}, \end{aligned} \tag{2.6}$$

where x_i is a formal local parameter and the central extension $\hat{\mathfrak{a}}_S^D$ is defined by the $\mathcal{O}_S$-valued 2-cocycle

$$c(A, B) := \sum_{i=1}^N \operatorname{Res}_{P_i} \langle A_i \mid B_i \rangle, \tag{2.7}$$

where $A = (A_i)_{i=1}^N, B = (B_i)_{i=1}^N \in \mathfrak{a}_S^D$

Example 2.2. For Example 2.1, the central extension $\hat{\mathfrak{g}}_S^{P_i}$ is naturally isomorphic to the space of $\hat{\mathfrak{g}}$-valued functions on S, $\hat{\mathfrak{g}}_S = \hat{\mathfrak{g}} \otimes_{\mathbb{C}} \mathcal{O}_S$, where $\hat{\mathfrak{g}}$ is the affine Lie algebra associated to $\mathfrak{g}$.

Let $\mathfrak{a}_{\dot{\mathfrak{X}}}^D$ be the space of meromorphic sections of $\mathfrak{a}_{\mathfrak{X}}$ which are globally defined along the fibre of $\pi_{\mathfrak{X}/S}$ and holomorphic except at D:

$$\mathfrak{a}_{\dot{\mathfrak{X}}}^D := (\pi_{\mathfrak{X}/S})_*(\mathfrak{a}_{\mathfrak{X}}(*D)), \tag{2.8}$$

which is naturally regarded as a Lie subalgebra of $\hat{\mathfrak{a}}_S^D$.

Example 2.3. For the Lie algebra bundle $\mathfrak{g}_{\mathfrak{X}}$ in Example 2.1, we denote $\mathfrak{a}_{\dot{\mathfrak{X}}}^D$ by $\mathfrak{g}_{\dot{\mathfrak{X}}}^D$. The section $w_{\alpha(H)}(t - z_i)e_\alpha$ (cf. (A.2)) belongs to $\mathfrak{g}_{\dot{\mathfrak{X}}}^D$.

Let us take $\hat{\mathfrak{a}}_S^{P_i}$-modules $\mathcal{M}_i$ $(i = 1, \dots, N)$ with the same level $\hat{k} = k$ and define $\mathcal{M} = \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_N$.

Definition 2.4. (i) The sheaf of *conformal blocks* $\mathcal{CB}(\mathfrak{a}_{\mathfrak{X}}, D, \mathcal{M})$ is defined to be the space of $\mathcal{O}_S$-linear functionals on $\mathcal{M}$ which vanishes on $\mathfrak{a}_{\dot{\mathfrak{X}}}^D \mathcal{M}$: $\Phi \in \mathcal{H}om_{\mathcal{O}_S}(\mathcal{M}, \mathcal{O}_S)$ belongs to $\mathcal{CB}(\mathfrak{a}_{\mathfrak{X}}, D, \mathcal{M})$ if and only if it satisfies

$$\Phi(A_{\dot{\mathfrak{X}}} v) = 0 \quad \text{for all } A_{\dot{\mathfrak{X}}} \in \mathfrak{a}_{\dot{\mathfrak{X}}}^D \text{ and } v \in \mathcal{M}. \tag{2.9}$$

This equation (2.9) is called the *Ward identity*.

(ii) There is a flat connection on $\mathcal{CB}(\mathfrak{a}_{\mathfrak{X}}, D, \mathcal{M})$. A flat section is called the *N-point function*. The set of N-point functions is denoted by $\mathcal{CB}^{\text{hor}}(\mathfrak{a}_{\mathfrak{X}}, D, \mathcal{M})$.

The flat connection on $\mathcal{CB}(\mathfrak{g}_{\mathfrak{X}}, D, \mathcal{M})$ is defined as follows. We use the notation,

$$\rho_i^*(h_r\{\theta_i(x_i)\}) = \sum_{m \in \mathbb{Z}} \theta_{i,m} \rho_i^*(h_r[m]), \tag{2.10}$$

for an element $\theta_i(x_i) = \sum_{m \in \mathbb{Z}} \theta_{i,m} x_i^m$, of $\mathcal{O}_S((x_i))$. The connection on $\mathcal{CB}(\mathfrak{g}_{\mathfrak{X}}, D, \mathcal{M})$ in the direction of ξ_r is defined by

$$\nabla^*_{\partial/\partial \xi_r} = \frac{\partial}{\partial \xi_r} - \rho^*(h_r\{Z(t)\}) := \frac{\partial}{\partial \xi_r} - \sum_{i=1}^N \rho_i^*(h_r\{Z(x_i + z_i)\}), \tag{2.11}$$

where $Z(t)$ is a meromorphic function with poles in D and has quasi-periodicity

$$Z(t+m\tau+n)=Z(t)-2\pi i m. \tag{2.12}$$

(In (2.11) $Z(t)$ is expanded around z_i in the power series of $x_i=t-z_i$ and substituted into (2.10).) For example, we can take $Z(t)=\zeta_{11}(t-z_1)$ (cf. (A.2)).

We denote $\mathcal{CB}(\mathfrak{g}_{\mathfrak{X}},D,\mathcal{M})$ and $\mathcal{CB}^{\mathrm{hor}}(\mathfrak{g}_{\mathfrak{X}},D,\mathcal{M})$ by $\mathcal{CB}(D,\mathcal{M})$ and $\mathcal{CB}^{\mathrm{hor}}(D,\mathcal{M})$ for short. We mean by the *WZW model* the theory on $\mathcal{CB}(D,\mathcal{M})$ or $\mathcal{CB}^{\mathrm{hor}}(D,\mathcal{M})$. Hereafter we assume that the $\hat{\mathfrak{g}}$-module M_i is generated over $\hat{\mathfrak{g}}$ by a $\mathfrak{g}$-submodule V_i on which the centre $\hat{k}\in\hat{\mathfrak{g}}$ acts as multiplication by k and $A\otimes x^m$ ($A\in\mathfrak{g}$, $m>0$) acts by 0. Put $\mathcal{M}_i:=M_i\otimes\mathcal{O}_S$, $V=V_1\otimes\cdots\otimes V_N$.

Lemma 2.5. *An N-point function $\Phi(H;v)$ of the WZW model is determined by its values on $v\in V(0)$, where $V(0)$ is the 0-weight space of V. In other words, the following restriction map is injective:*

$$\mathcal{CB}^{\mathrm{hor}}(D,\mathcal{M})\ni\Phi\mapsto\Phi(H;v)\in V^*(0)\otimes\mathcal{O}_S. \tag{2.13}$$

This is a consequence of the Ward identity (2.9) and the flatness condition.

Let us return to the discussion for the general Lie algebra bundle $\mathfrak{a}_{\mathfrak{X}}$. Let Q be a point not contained in D and $\mathrm{V}ac_{Q,k}$ be the $\hat{\mathfrak{a}}^Q_S$-module induced from the trivial $\hat{\mathfrak{a}}^Q_{S,+}$-module $\mathcal{O}_{S,k}$ of level k:

$$\mathrm{V}ac_{Q,k}:=\mathrm{Ind}_{\hat{\mathfrak{a}}^Q_{S,+}}^{\hat{\mathfrak{a}}^Q_S}\mathcal{O}_{S,k} \tag{2.14}$$

where $\mathcal{O}_{S,k}=\mathcal{O}_S|0\rangle$ as a linear space, $\mathfrak{a}^Q_{S,+}\mathcal{O}_{S,k}=0$ and $\hat{k}$ acts as a multiplication by k. We call $\mathrm{V}ac_{Q,k}$ the *vacuum module* of level k at Q.

Lemma 2.6. *The canonical linear map $\mathcal{M}\ni v\mapsto v\otimes|0\rangle\in\mathcal{M}\otimes\mathrm{V}ac_{Q,k}$ induces isomorphisms:*

$$\begin{aligned}\mathcal{CB}(\mathfrak{a}_{\mathfrak{X}},D,\mathcal{M})&\xleftarrow{\sim}\mathcal{CB}(\mathfrak{a}_{\mathfrak{X}},D+Q,\mathcal{M}\otimes\mathrm{V}ac_{Q,k}),\\ \mathcal{CB}^{\mathrm{hor}}(\mathfrak{a}_{\mathfrak{X}},D,\mathcal{M})&\xleftarrow{\sim}\mathcal{CB}^{\mathrm{hor}}(\mathfrak{a}_{\mathfrak{X}},D+Q,\mathcal{M}\otimes\mathrm{V}ac_{Q,k}).\end{aligned} \tag{2.15}$$

This property is called *propagation of vacua* in [TUY].

The following proposition reveals the real nature of the operator $\hat{\tau}(u)$ in (1.5). The operator obtained directly from the WZW model differs from $\hat{\tau}(u)$ by conjugation. Let us define an operator $\tilde{\tau}(u)$ by

(2.16)

$$\begin{aligned}\tilde{\tau}(u)&=\Pi(H,\tau)\hat{\tau}(u)\Pi(H,\tau)^{-1}\\ &=\frac{1}{2}\sum_{r=1}^{l}\nabla^2_{\xi_r,u}+\sum_{r=1}^{l}\frac{\partial}{\partial\xi_r}\log\Pi(H,\tau)\nabla_{\xi_r,u}\\ &\quad+\frac{1}{2}\sum_{i,j=1}^{N}\sum_{\alpha\in\Delta}w_{\alpha(H)}(z_i-u)w_{-\alpha(H)}(z_j-u)\rho^*_j(e_{-\alpha})\rho^*_i(e_\alpha)+2\pi i h^\vee\frac{\partial}{\partial\tau}\log\Pi(H,\tau),\end{aligned}$$

where $\Pi(H,\tau)$ is the normalised Weyl-Kac denominator,

$$\Pi(H,\tau)=q^{\dim\mathfrak{g}/24}(q;q)^l_\infty\prod_{\alpha\in\Delta_+}(e^{\pi i\alpha(H)}-e^{-i\pi\alpha(H)})\prod_{\alpha\in\Delta}(qe^{2\pi i\alpha(H)};q)_\infty. \tag{2.17}$$

Here we use the usual notations, $q=e^{2\pi i\tau}$ and $(x;q)_\infty=\prod_{n=0}^\infty(1-xq^n)$.

Proposition 2.7. *According to Lemma 2.6, there is a $(N+1)$-point function $\tilde{\Phi}$ corresponding to $\Phi \in \mathcal{CB}^{\mathrm{hor}}(D, \mathcal{M})$. We have*

$$(\tilde{\tau}(u)\Phi)(H; v) = \tilde{\Phi}(H; v \otimes S[-2]|0\rangle), \tag{2.18}$$

for $H \in S$, $v \in V(0)$, where u is the coordinate of Q on the complex plane and $S[-2]$ is defined as a coefficient of the Sugawara tensor*,*

$$S(u) := \frac{1}{2} \sum_{p=1}^{\dim \mathfrak{g}} {}^{\circ}_{\circ} J_p(u) J_p(u) {}^{\circ}_{\circ} = \sum_{n \in \mathbb{Z}} S[n] z^{-n-1}. \tag{2.19}$$

Here $\{J_p\}_{p=1,\dots,\dim \mathfrak{g}}$ is an orthonormal basis of $\mathfrak{g}$ and the symbol ${}^{\circ}_{\circ}\ {}^{\circ}_{\circ}$ is the normal ordering operation.

This means that $\tilde{\tau}(u)\Phi(v_1 \otimes \cdots \otimes v_N)$ is the correlation function $\langle S(u) v_1(z_1) \cdots v_N(z_n)\rangle$ of $S(u)$ in the context of the conformal field theory.

Hitherto the level is arbitrary. To prove Theorem 1.1, we need to fix the level to the critical value, $k = -h^\vee$, where $S[-2]|0\rangle$ is a singular vector of imaginary weight. Roughly speaking, by virtue of this fact the correlation function of two Sugawara tensors, $\langle S(u) S(u') v_1(z_1) \cdots v_N(z_n)\rangle$, is irrelevant to the order of insertion of $S(u)$ and $S(u')$, from which the commutativity (1.7) follows.

3. Wakimoto modules at the critical level

The Bethe vector (1.10) is constructed by means of the Wakimoto realisation of affine Lie algebras from the free field theory. In this section we review basic facts about the Wakimoto representations, following [K]. See also [W], [FF1], [FF2], [FFR].

The *bosonic ghost fields*,

$$\beta_\alpha(z) = \sum_{m \in \mathbb{Z}} z^{-m-1} \beta_\alpha[m], \qquad \gamma^\alpha(z) = \sum_{m \in \mathbb{Z}} z^{-m} \gamma^\alpha[m], \qquad (\alpha \in \Delta_+) \tag{3.1}$$

satisfy the following operator product expansions:

$$\beta_\alpha(z) \gamma^{\alpha'}(w) \sim \frac{\delta_\alpha^{\alpha'}}{z - w}. \tag{3.2}$$

We denote the Heisenberg algebra generated by $\beta_\alpha[m]$ and $\gamma^\alpha[m]$ ($\alpha \in \Delta_+$, $m \in \mathbb{Z}$) by $\widehat{\mathrm{Gh}}(\mathfrak{g})$.

The *ghost Fock space* $\mathcal{F}^{\mathrm{gh}}$ is defined as a left $\widehat{\mathrm{Gh}}(\mathfrak{g})$-module generated by the vacuum vector $|0\rangle^{\mathrm{gh}}$, satisfying

$$\beta_\alpha[m]|0\rangle^{\mathrm{gh}} = 0, \qquad \gamma^\alpha[n]|0\rangle^{\mathrm{gh}} = 0 \tag{3.3}$$

for any $\alpha \in \Delta_+$, $m \geqq 0$, $n > 0$. The *normal ordered product* $:P:$ of a monomial P of $\beta_\alpha[m]$'s and $\gamma^\alpha[m]$'s is defined by putting annihilation operators of $|0\rangle^{\mathrm{gh}}$ appearing in P to the right side in the product.

The *free boson fields*,

$$\begin{aligned} &\phi_i(z) := \phi_i[0]\log z + \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{z^{-m}}{-m}\phi_i[m], \qquad \partial\phi_i(z) := \sum_{m\in\mathbb{Z}} z^{-m-1}\phi_i[m], \\ &\phi(H;z) = \sum_{i=1}^{l} a_i\phi_i(z), \qquad \phi[H;m] = \sum_{i=1}^{l} a_i\phi_i[m], \end{aligned} \tag{3.4}$$

for $H = \sum_{i=1}^{l} a_i H_i \in \mathfrak{h}$ have trivial operator product expansions:

$$\phi(H;z)\phi(H';w) \sim 0, \qquad \partial\phi(H;z)\partial\phi(H';w) \sim 0, \tag{3.5}$$

for any $H, H' \in \mathfrak{h}$.

The commutative algebra generated by $\phi_i[m]$ ($i = 1, \dots, l$, $m \in \mathbb{Z}$) is denoted by $\widehat{\mathrm{Bos}}(\mathfrak{g})$.

For any one-form $\lambda(x)dx \in \mathfrak{h}^* \otimes \mathbb{C}((x))dx$, we define a one-dimensional representation $\sigma_{\lambda(x)dx}$ of $\widehat{\mathrm{Bos}}(\mathfrak{g})$ by

$$\begin{aligned} \sigma_{\lambda(x)dx} &= \mathbb{C}|\lambda(x)dx\rangle, \\ f(x)\,|\lambda(x)dx\rangle &= \mathrm{Res}_{x=0}(\lambda(x), f(x))\,dx, \end{aligned} \tag{3.6}$$

where $f(x) \in \mathfrak{h} \otimes \mathbb{C}((x))$ is identified with an element of $\widehat{\mathrm{Bos}}$ by the isomorphism defined by $H_i \otimes x^m \mapsto H_i[m]$ and $(\cdot,\cdot)$ is the canonical pairing of $\mathfrak{h}^*$ and $\mathfrak{h}$. In other words,

$$\phi[H;m]\,|\lambda(x)dx\rangle = \lambda^{(-m-1)}(H)\,|\lambda(x)dx\rangle, \tag{3.7}$$

where $\lambda(x)dx = \sum_{n\in\mathbb{Z}} \lambda^{(n)} x^n\,dx$, $\lambda^{(n)} \in \mathfrak{h}^*$. Hereafter we assume that $\lambda^{(n)} = 0$ for $n \leqq -2$.

Proposition 3.1. [W], [FF1], [FF2], [K]. *For each Chevalley generator H_i, E_i or F_i of $\mathfrak{g}$, there exists a differential polynomial of the free fields,*

$$X(z) = \sum_{m\in\mathbb{Z}} X[m]z^{-m-1} := {:}R(X;\gamma(z),\beta(z),\partial\phi(z)){:},$$

which gives the corresponding Kac-Moody current. Namely, a Lie algebra homomorphism ω from the affine Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}] \oplus \mathbb{C}\hat{k}$ to a completion of $\widehat{\mathrm{Gh}}(\mathfrak{g}) \otimes \widehat{\mathrm{Bos}}(\mathfrak{g})$ can be defined by

$$\omega(X\otimes t^m) = X[m], \qquad \omega(\hat{k}) = -h^\vee, \tag{3.8}$$

for all $X \in \mathfrak{g}$, $m \in \mathbb{Z}$, where $\hat{k}$ is the centre of $\hat{\mathfrak{g}}$ and $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Moreover, the Sugawara tensor $S(z)$ defined by (2.19) *is expressed in terms of the free bosons as follows:*

$$S(z) := \frac{1}{2}\sum_{r=1}^{l} {:}\partial\phi(h_r;z)\partial\phi(h_r;z){:} - \frac{1}{2}\partial^2\phi(2\rho;z), \tag{3.9}$$

Definition 3.2. Denote $\mathcal{F}^{\mathrm{gh}} \otimes \sigma_{\lambda(x)dx}$ by $\mathrm{W}ak_{\lambda(x)dx}$. We regard this module as a $\hat{\mathfrak{g}}$-module through the above homomorphism $\omega : \hat{\mathfrak{g}} \to \widehat{\mathrm{Gh}}(\mathfrak{g}) \otimes \widehat{\mathrm{Bos}}(\mathfrak{g})$ and call it a *Wakimoto module* of level $-h^\vee$ (or of critical level) with highest weight $\lambda(x)dx$.

The Wakimoto module contains a $\mathfrak{g}$-submodule isomorphic to the dual Verma module M_λ^* of $\mathfrak{g}$ with the highest weight $\lambda = \lambda^{(-1)}$. (See Proposition 4.4 of [K].) We denote it by $\mathrm{W}ak^0_{\lambda(x)dx}$. It satisfies for any $m > 0$ and $X \in \mathfrak{g}$,

$$X[m]\mathrm{W}ak^0_{\lambda(x)dx} = 0. \tag{3.10}$$

4. Free field theories and Bethe vectors

We can "decompose" the WZW model in Section 2 into free field theories, using the Wakimoto realisation, Theorem 3.1. The Bethe vectors in Theorem 1.2 are nothing but the N-point functions of the free field theories.

Let us define free field theories in the framework introduced in Section 2. We defined the Lie algebra bundle $\mathfrak{g}_{\mathfrak{X}}$ as a quotient of $S \times \mathbb{C} \times \mathfrak{g}$ by the $\mathbb{Z}^2$-action (2.5). Note that this action preserves the triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. We regard *vector* bundles,

$$\boldsymbol{\beta}_{\mathfrak{X}} := \mathbb{Z}^2 \backslash S \times \mathbb{C} \times \mathfrak{n}_+ \tag{4.1}$$

$$\boldsymbol{\gamma}_{\mathfrak{X}} := (\mathbb{Z}^2 \backslash S \times \mathbb{C} \times \mathfrak{n}_-) \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/S}, \tag{4.2}$$

$$\mathrm{Bos}_{\mathfrak{X}} := \mathbb{Z}^2 \backslash S \times \mathbb{C} \times \mathfrak{h}, \tag{4.3}$$

as abelian Lie algebra bundles and put $\mathrm{Gh}_{\mathfrak{X}} := \boldsymbol{\beta}_{\mathfrak{X}} \oplus \boldsymbol{\gamma}_{\mathfrak{X}}$. We apply Definition 2.4 to $\mathfrak{a}_{\mathfrak{X}} = \mathrm{Gh}_{\mathfrak{X}}$ and $\mathfrak{a}_{\mathfrak{X}} = \mathrm{Bos}_{\mathfrak{X}}$. The pairing (2.2) for $\mathrm{Bos}_{\mathfrak{X}}$ is trivial (i.e., $\langle \cdot \mid \cdot \rangle = 0$) and the pairing for $\mathrm{Gh}_{\mathfrak{X}}$ is defined by

$$\langle (A_1, B_1 dt) \mid (A_2, B_2 dt) \rangle^{\mathrm{gh}} := (A_1 \mid B_2)dt - (B_1 \mid A_2)dt \in \Omega^1_{\mathfrak{X}}, \tag{4.4}$$

where $A_i \in \boldsymbol{\beta}_{\mathfrak{X}}$, $B_i dt \in \boldsymbol{\gamma}_{\mathfrak{X}}$ for $i = 1, 2$ and $(\cdot|\cdot)$ denotes the inner product defined by (1.2). Here we identify $\boldsymbol{\beta}_{\mathfrak{X}}$ and $\boldsymbol{\gamma}_{\mathfrak{X}}$ with a subbundle of $\mathfrak{g}_{\mathfrak{X}}$ and a subbundle of $\mathfrak{g}_{\mathfrak{X}} \otimes \Omega^1_{\mathfrak{X}/S}$ respectively.

The algebra defined by (2.6) for $D = P$ (a point) in this case is isomorphic to $\widehat{\mathrm{Gh}}_S(\mathfrak{g}) = \widehat{\mathrm{Gh}}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathcal{O}_S$ when $\mathfrak{a}_{\mathfrak{X}} = \mathrm{Gh}_{\mathfrak{X}}$ and to $\widehat{\mathrm{Bos}}_S(\mathfrak{g}) = \widehat{\mathrm{Bos}}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathcal{O}_S$ when $\mathfrak{a}_{\mathfrak{X}} = \mathrm{Bos}_{\mathfrak{X}}$.

We denote the sheaf of conformal blocks $\mathcal{CB}(\mathfrak{a}_{\mathfrak{X}}, D, \mathcal{M})$ and the space of N-point functions $\mathcal{CB}^{\mathrm{hor}}(\mathfrak{a}_{\mathfrak{X}}, D, \mathcal{M})$ defined in Definition 2.4 for $\mathfrak{a}_{\mathfrak{X}} = \mathrm{Gh}_{\mathfrak{X}}, \mathrm{Bos}_{\mathfrak{X}}$ by

$$\mathcal{CB}^{\mathrm{gh}}(D, \mathcal{M}) := \mathcal{CB}(\mathrm{Gh}_{\mathfrak{X}}, D, M), \qquad \mathcal{CB}^{\mathrm{bos}}(D, \mathcal{M}) := \mathcal{CB}(\mathrm{Bos}_{\mathfrak{X}}, D, M), \tag{4.5}$$

$$\mathcal{CB}^{\mathrm{gh,hor}}(D, \mathcal{M}) := \mathcal{CB}^{\mathrm{hor}}(\mathrm{Gh}_{\mathfrak{X}}, D, M), \quad \mathcal{CB}^{\mathrm{bos,hor}}(D, \mathcal{M}) := \mathcal{CB}^{\mathrm{hor}}(\mathrm{Bos}_{\mathfrak{X}}, D, M), \tag{4.6}$$

and define

$$\begin{aligned} \mathcal{CB}^{\mathrm{free}}(D, \mathcal{M}) &:= \mathcal{CB}(\mathrm{Gh}_{\mathfrak{X}} \oplus \mathrm{Bos}_{\mathfrak{X}}, D, \mathcal{M}), \\ \mathcal{CB}^{\mathrm{free,hor}}(D, \mathcal{M}) &:= \mathcal{CB}^{\mathrm{hor}}(\mathrm{Gh}_{\mathfrak{X}} \oplus \mathrm{Bos}_{\mathfrak{X}}, D, \mathcal{M}). \end{aligned} \tag{4.7}$$

Assume that $\mathcal{M}_i$ is a $\widehat{\mathrm{Gh}}_S^{P_i}$-module and $\mathcal{N}_i$ is a $\widehat{\mathrm{Bos}}_S^{P_i}$-module. Then $\mathcal{M}_i \otimes_{\mathcal{O}_S} \mathcal{N}_i$ is a $(\widehat{\mathrm{Gh}}_S^{P_i} \oplus \widehat{\mathrm{Bos}}_S^{P_i})$-module and hence is a $(\mathrm{Gh}_{\mathfrak{X}} \oplus \mathrm{Bos}_{\mathfrak{X}})_S^{\wedge, P_i}$-module. The important fact is that the N-point function of the free field theory naturally gives the N-point function of the WZW model.

Proposition 4.1. $\mathfrak{g}^D_{\mathfrak{X}}(\bigotimes(\mathcal{M}_i \otimes_{\mathcal{O}_S} \mathcal{N}_i)) \subset (\mathrm{Gh}^D_{\mathfrak{X}} \oplus \mathrm{Bos}^D_{\mathfrak{X}})(\bigotimes(\mathcal{M}_i \otimes_{\mathcal{O}_S} \mathcal{N}_i))$. *Hence the identity morphism from* $\bigotimes(\mathcal{M}_i \otimes_{\mathcal{O}_S} \mathcal{N}_i)$ *to itself induces an* $\mathcal{O}_S$*-linear map:*

$$\iota : \mathcal{CB}^{\mathrm{free}}(D, \bigotimes(\mathcal{M}_i \otimes_{\mathcal{O}_S} \mathcal{N}_i)) \to \mathcal{CB}(D, \bigotimes(\mathcal{M}_i \otimes_{\mathcal{O}_S} \mathcal{N}_i)). \tag{4.8}$$

Moreover, this induces a $\mathbb{C}$*-linear map between spaces of* N*-point functions:*

$$\iota : \mathcal{CB}^{\mathrm{free,hor}}(D, \bigotimes(\mathcal{M}_i \otimes_{\mathcal{O}_S} \mathcal{N}_i)) \to \mathcal{CB}^{\mathrm{hor}}(D, \bigotimes(\mathcal{M}_i \otimes_{\mathcal{O}_S} \mathcal{N}_i)), \tag{4.9}$$

If $\mathcal{M}_i = \mathcal{F}^{\mathrm{gh}} \otimes \mathcal{O}_S$ and $\mathcal{N}_i = \sigma_{\mu_i(x)dx} \otimes \mathcal{O}_S$ for $\mu_i(x)dx \in \mathfrak{h}^*((x))dx$, then an N-point function of the free field gives an N-point function of the WZW model with the Wakimoto modules. It is easy to see that

$$\mathcal{CB}^{\mathrm{free,hor}}(D, \bigotimes(\mathcal{M}_i \otimes_{\mathcal{O}_S} \mathcal{N}_i)) \cong \mathcal{CB}^{\mathrm{gh,hor}}(D, \mathcal{M}) \otimes_{\mathbb{C}} \mathcal{CB}^{\mathrm{bos,hor}}(D, \mathcal{N}), \tag{4.10}$$

hence it is enough to find N-point functions in $\mathcal{CB}^{\mathrm{gh,hor}}(D, (\mathcal{F}^{\mathrm{gh}})^{\otimes N})$ and $\mathcal{CB}^{\mathrm{bos,hor}}(D, \bigotimes_{i=1}^N \sigma_{\mu_i(x)dx} \otimes \mathcal{O}_S)$ to construct an N-point function of the WZW model. Let us denote $\boldsymbol{\sigma}_{\vec{\mu}dx} := \bigotimes_{i=1}^N \sigma_{\mu_i(x)dx} \otimes \mathcal{O}_S$ and $|\vec{\mu}dx\rangle := \bigotimes_{i=1}^N |\mu_i(x)dx\rangle$ for simplicity.

Lemma 4.2. *(i)* $\mathcal{CB}^{\mathrm{gh,hor}}(D, (\mathcal{F}^{\mathrm{gh}})^{\otimes N}) = \mathbb{C}\Phi^{\mathrm{gh}}(H; v)$, *where*

$$\Phi^{\mathrm{gh}}(H; v) = \Pi(H, \tau)^{-1}(\text{coefficient of } (|0\rangle^{\mathrm{gh}})^{\otimes N} \text{ in } v).$$

(ii) $\mathcal{CB}^{\mathrm{bos,hor}}(D, \boldsymbol{\sigma}_{\vec{\mu}dx})$ *is one-dimensional if and only if there exists* $\mu(t)dt \in (\pi_{\mathfrak{X}/S})_*(\mathfrak{h}^* \otimes \Omega^1_{\mathfrak{X}}(*D))$ *such that each* $\mu_i(x_i)dx_i$ *is a Laurent expansion of* $\mu(t)dt$ *at* $t = z_i$ *with respect to* $x_i = t - z_i$*. In this case,* $\mathcal{CB}^{\mathrm{bos,hor}}(D, \boldsymbol{\sigma}_{\vec{\mu}dx}) = \mathbb{C}\Phi^{\mathrm{bos}}(v)$, *where*

$$\Phi^{\mathrm{bos}}(v) = (\text{coefficient of } |\vec{\mu}dx\rangle \text{ in } v).$$

Otherwise, $\mathcal{CB}^{\mathrm{bos}}(D, \boldsymbol{\sigma}_{\vec{\mu}dx}) = 0$.

Assume λ_i $(i = 1, \dots, N)$ satisfies the condition (1.8), and put $\mu_i = \lambda_i$ $(i = 1, \dots, N)$, $\mu_{N+j} = -\alpha_{i(j)}$ $(j = 1, \dots, M)$. Lemma 4.2 guarantees that $\mathcal{CB}^{\mathrm{free,hor}}(D + D', \boldsymbol{\sigma}_{\vec{\mu}dx})$ is one-dimensional if we define $D = P_1 + \cdots + P_N$ (P_i has the coordinate z_i), $D' = Q_1 + \cdots + Q_M$ (Q_j has the coordinate t_j) and $\mu_i(x)dx$ $(i = 1, \dots, N + M)$ as the Laurent expansion of $\mu(t)dt$ defined by

$$\mu(t)dt = \sum_{i=1}^N \lambda_i \zeta_{11}(t - z_i) - \sum_{j=1}^M \alpha_{i(j)} \zeta_{11}(t - t_j). \tag{4.11}$$

The basis of this one-dimensional space is $\Phi^{\mathrm{free}}(H; v) = \Phi^{\mathrm{gh}}(H; v)\Phi^{\mathrm{bos}}(v)$.

We assume that the parameters z_i $(i = 1, \dots, N)$ and t_j $(j = 1, \dots, M)$ satisy the Bethe Ansatz equations (1.9). Then by Lemma 2 of [FFR] (or by Corollary 5.2 of [K]), the *screening vector* scr_j (cf. (5.21) of [K]) in $\mathrm{W}ak_{\mu_{N+j}(x)dx}$ is a singular vector of imaginary weight. Thanks to this property and (3.10), the linear functional defined by

$$\Psi(H; v) := \Phi^{\mathrm{free}}(H; v \otimes scr_1 \otimes \cdots \otimes scr_M) \tag{4.12}$$

for $v \in \bigotimes_{i=1}^N \mathrm{W}ak^0_{\mu_i(x)dx} \cong \bigotimes_{i=1}^N M^*_{\lambda_i}$ has the same property as (2.18):

$$\tilde{\Psi}(H; v \otimes S[-2]|0\rangle) = \tilde{\tau}(u)\Psi(H; v). \tag{4.13}$$

On the other hand, we can compute the left hand side of (4.13), using the expression (3.9) and the Ward identity (2.9) for the free boson. The result is

$$\tilde{\Psi}(H; v \otimes S[-2]|0\rangle) = \tau_\Psi(u)\Psi(H; v), \tag{4.14}$$

namely $\Psi(H; v)$ is the eigenvector of $\hat{\tau}(u)$ with the eigenvalue $\tau_\Psi(u)$ defined by (1.12).

The explicit form of $\Psi(H; v)$, (1.10), is derived by the same argument as that of the rational case. See Lemma 3 of [FFR].

Appendix A. Elliptic functions

We follow the notations of [M] for theta functions and denote the odd theta function by

$$\theta_{11}(z, \tau) = \sum_{n\in\mathbb{Z}} \exp\left(\pi i \tau \left(n + \frac{1}{2}\right)^2 + 2\pi i \left(z + \frac{1}{2}\right)\left(n + \frac{1}{2}\right)\right). \tag{A.1}$$

We use a multiplicatively and additively quasi-periodic function,

$$w_c(z) := \frac{\theta'_{11}(0)\theta_{11}(z-c)}{\theta_{11}(z)\theta_{11}(-c)}, \qquad \zeta_{11}(z) := \frac{d}{dz}\log\theta_{11}(z). \tag{A.2}$$

These functions are characterised by the properties

(A.3)

$$w_c(z+1) = w_c(z), \qquad w_c(z+\tau) = e^{2\pi i c} w_c(z), \qquad w_c(z) \sim z^{-1} \text{ around } z = 0.$$

(A.4)

$$\zeta_{11}(z+1) = \zeta_{11}(z), \qquad \zeta_{11}(z+\tau) = \zeta_{11}(z) - 2\pi i, \qquad \zeta_{11}(z) \sim z^{-1} \text{ around } z = 0.$$

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